Analytical Statistical Reconstruction Algorithm with the Direct Use of Projections Performed in Spiral Cone-beam Scanners

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***Abstract*—This paper is concerned with the originally formulated 3D statistical model-based iterative reconstruction algorithm for spiral cone-beam x-ray tomography. The conception proposed here is based on a formulation of the reconstruction problem as a shift invariant system. This algorithm significantly improves the quality of the subsequently reconstructed images, so allowing a decrease in the x-ray dose absorbed by a patient. The analytical roots of the algorithm proposed here permit a decrease in the complexity of the reconstruction problem in comparison with other model- based iterative approaches. In this paper, we proved that this statistical approach, originally formulated for parallel beam geometry, can be adapted for helical cone-beam geometry of scanner, with the direct use of projections. Computer simulations have shown that the reconstruction algorithm presented here outperforms conventional analytical methods with regard to the image quality obtained.**

I. INTRODUCTION

Nowadays, the most significant problem in medical computer tomography is the development of image reconstruction algorithms from projections which would enable the reduction of the impact of measurement noise on the quality of tomography images. This kind of approach is intended to improve image quality and, in consequence, reduce the dose of X-ray radiation while at the same time preserving an appropriate level of quality in the tomography images. The concept has found its application in the form of statistical reconstruction algorithms. One of the most interesting from the scietific and practical point of view, an approach, called MBIR (Model-Based Iterative Reconstruction), is presented in publications like [1], where a probabilistic model of the measurement signals is described analytically. The objective in those solutions was devised according to an algebraic scheme of the signal processing for formulating the reconstruction problem [2]. An algebraic scheme has been selected in this case for one very obvious reason - the measurement noise can be modelled relatively easily, because each measurement is considered separately. Such a scheme adds significant calculative complexity to the problem. The time for image reconstruction becomes difficut from the practical point of view. For instance, if the image resolution is assumed to be I×I pixels, the complexity of the algebraic problem is of the level of $\mathrm{I}\times\mathrm{I}\times number\_of\_measurements \times number\_of\_cross-sections$ (in 3D tomography); a multiple of I to the power of four in total.

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The difficulties mentioned above connected with the use of an algebraic methodology can be decreased by using an analytical strategy of reconstructed image processing. In previous papers we have shown how to formulate the analytical reconstruction problem consistent with the ML methodology for parallel scanner geometry [4]. This strategy has been used for fan-beams [3], and finally for the spiral cone-beam scanner [5]. However, an approach to the reformulation of the reconstruction problem from parallel to real scanner geometries, called rebinning, was applied there. Much more popular 3D reconstruction methods, which are implemented in practice, are FDK (Feldkamp)-type algorithms that use projections obtained from spiral cone-beam scanners directly (see e.g. [6]). In this paper, we present a mathematical derivation of a method for the direct (i.e. without rebinning) adaptation of spiral cone-beam projections to the statistical analytical reconstruction algorithm originally formulated by us.

II. ADAPTATION OF THE 2D ANALYTICAL APPROXIMATE RECONSTRUCTION PROBLEM TO SPIRAL CONE-BEAM PROJECTIONS

A foundation for our conception of the model-based iterative statistical algorithm is the 2D analytical approximate reconstruction problem which was originally formulated for a parallel scanner geometry [3] of scanner, as follows:

$$\mu_{\min} = \arg\min_{\mu} \left(\frac{n_0}{2} \sum_{i=1}^{\mathrm{I}} \sum_{j=1}^{\mathrm{J}} \cdot \right. \tag{1}$$
$$\left. \cdot \left(\sum_{\bar{i}} \sum_{\bar{j}} \mu\left(x_{\bar{i}}, y_{\bar{j}}\right) \cdot h_{\Delta i, \Delta j} - \tilde{\mu}\left(x_i, y_j\right) \right)^2 \right),$$

where coefficients $h_{\Delta i,\Delta j}$ are precaculated in the numerical way according to the following relation:

$$h_{\Delta i,\Delta j} = \Delta_\alpha \sum_{\psi=0}^{\Psi-1} int\left(\Delta i \cos\psi\Delta_\alpha + \Delta j \sin\psi\Delta_\alpha\right), \tag{2}$$

and $\tilde{\mu}\left(i,j\right)$ is an image obtained by way of a back-projection operation; $int\left(\Delta s\right)$ is an interpolation function used in the back-projection operation; every projection is carried out after a rotation by Δ_α.

Above presented shift-invariant system is much better conditioned than quadratic form used in algebraic aproaches [7], and can be a starting point for the design of a 3D iterative reconstruction algorithm for spiral cone-beam scanner geometry. One of the principal reconstruction methods devised for the cone-beam spiral scanner is the generalized FDK algorithm. In the traditional FDK approach, the cone-beam projections are filtered and then back-projected in three dimensions. This methodology is adapted to our original iterative model-based reconstruction concept.

Taking into consideration the definition of the two-dimensional inverse Fourier transform, and the frequential form of the relation between the original image of a cross-section of an examined object represented by function $\mu(x, y)$ and the image obtained after the back-projection operation $\tilde{\mu}(x, y)$, we obtain:

$$\tilde{\mu}(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{|f|} M(f_1, f_2)\, e^{j2\pi(f_1 x + f_2 y)} df_1 df_2, \quad (3)$$

which, after converting to polar coordinates and using the projection slice theorem (taking into account a full revolution of the projection system), takes the form:

$$\tilde{\mu}(x, y) = \frac{1}{2} \int_{-\pi}^{\pi} \int_{-\infty}^{\infty} \bar{P}(f, \alpha^p)\, e^{j2\pi f(x\cos\alpha^p + y\sin\alpha^p)} df d\alpha^p. \quad (4)$$

Then, after transferring the projections into the spatial domain, and arranging the right hand side of the formula and changing the order of integration, we getwe have the formula:

$$\tilde{\mu}(x, y) = \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\pi}^{\pi} \bar{p}^p(s, \alpha^p)\, e^{j2\pi f(x\cos\alpha^p + y\sin\alpha^p - s)} d\alpha^p ds df, \quad (5)$$

where $\bar{p}^p(s, \alpha^p)$ are projections obtained in a hypothetical parallel scanner (after interpolation).

Next, after converting the attenuation function into polar coordinates, we obtain:

$$\tilde{\mu}(r\cos\phi, r\sin\phi) = \quad (6)$$
$$\frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\pi}^{\pi} \bar{p}^p(s, \alpha^p)\, e^{j2\pi f[r\cos(\alpha^p - \phi) - s]} d\alpha^p ds df.$$

In our considerations, we should also take into account the application of the interpolation function used during the back-projection operation, which should be placed appropriately (a frequency representation of this function) in the formula above, as follows:

$$\breve{\mu}(x, y) = \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\pi}^{\pi} \quad (7)$$
$$INT(f)\, p^p(s, \alpha^p)\, e^{j2\pi f[r\cos(\alpha^p - \phi) - s]} d\alpha^p ds df.$$

After suitable transformation we obtain a relationship for the fan-beam image reconstruction method:

$$\breve{\mu}(x, y) = \frac{\mathrm{R}_f}{2} \int_{0}^{2\pi} \int_{-\beta_m}^{\beta_m} p^f(\beta, \alpha^f) \quad (8)$$
$$\frac{\cos\beta}{\dot{u}^2} \int_{-\infty}^{\infty} INT(f)\, e^{j2\pi f \dot{u}\sin(\dot{\beta} - \beta)} df d\beta d\alpha^f,$$

where $p^f(\beta, \alpha^f)$ are projections obtained in a hypothetical fan-beam scanner, and

$$\dot{u} = \left(x\cos\alpha^f + y\sin\alpha^f\right)^2 + \left(R_f + x\sin\alpha^f - y\cos\alpha^f\right)^2. \quad (9)$$

There is a several serious drawbacks associated with the use of the fan-beam reconstruction method formulated like this. It stems from the dependence of equation (9) on the parameter $\dot{u}$, which poses certain practical problems when carrying out the calculations during the reconstruction process. Instead of a simple formula for the convolution kernel, it now becomes necessary to determine a different form of the kernel for every point of the object's cross-section. This is because $\dot{u}$ represents the distance of the point (r, ϕ) from the radiation source. Therefore, by changing the angle α^f, we also change $\dot{u}$. The appropriate adjustment is based on a term in equation (9), which is reproduced here in a suitably amended form:

$$int(s) = \int_{-\infty}^{\infty} INT(f)\, e^{j2\pi f \dot{u}\sin(\dot{\beta} - \beta)} df. \quad (10)$$

In this equation, the integration is carried out with respect to the frequency f. The next step will be to make a substitution for f, using the following expression:

$$f^f = \frac{f \cdot \dot{u} \cdot \sin\beta}{\mathrm{R}_f \cdot \beta}. \quad (11)$$

If at the same time we change the limits of integration, the convolving function will be modified to:

$$int^f(\beta) = \frac{\mathrm{R}_f \cdot \beta}{\dot{u} \cdot \sin\beta} \int_{-\infty}^{\infty} INT\left(\frac{f^f \cdot f_0}{f_0^f}\right) e^{j2\pi f^f \mathrm{R}_f \beta} df^f, \quad (12)$$

where

$$f_0^f = \frac{f_0 \cdot \dot{u} \cdot \sin\beta}{\mathrm{R}_f \cdot \beta}. \quad (13)$$

Unfortunately, even here we encounter problems caused by the dependence of the cut-off frequency f_0^f on the parameter $\dot{u}$. On the other hand, if we were to establish a constant value for f_0^f it would mean that the reconstruction process for the point (r, ϕ) would have a different resolution (determined by the value of the cut-off frequency f_0) for every angle α^f. However, if we put aside the assumption of uniform resolution for the resulting reconstructed image, then, by manipulating the values $\dot{u}$ and f_0, the varying value of f_0^f can be fixed as:

$$f_0^f = \mathrm{f}_0^f = \frac{1}{\mathrm{R}_f \cdot \Delta_\beta}. \tag{14}$$

Let us assume that we apply a linear interpolation function in formula (8). The frequency form of the linear interpolation function is given by this formula:

$$INT_L(f) = \frac{\sin^2(\pi f \Delta_s)}{(\pi f \Delta_s)^2}. \tag{15}$$

Taking into account in the formula (12) the proposed interpolation function given by (15), we obtain the following relation:

$$int_L^f(\beta) = \frac{\mathrm{R}_f \cdot \beta}{\dot{u} \cdot \sin\beta} \begin{cases} \frac{1}{\Delta'_s}\left(1 - \frac{\mathrm{R}_f|\beta|}{\Delta'_s}\right) & \text{for} \quad |\beta| \le \Delta'_s \\ 0 & \text{for} \quad |\beta| \ge \Delta'_s \end{cases}, \tag{16}$$

where $\Delta'_s = f_0/f_0^f$, and next, bearing in mind relations (14), it leads immediately to:

$$int_L^f(\beta) = \frac{\beta}{\dot{u} \cdot \sin\beta} \begin{cases} \frac{\Delta_s}{\Delta_\beta}\left(1 - \frac{\Delta_s|\beta|}{\Delta_\beta}\right) & \text{for} \quad |\beta| \le \frac{\Delta_\beta}{\Delta_s} \\ 0 & \text{for} \quad |\beta| \ge \frac{\Delta_\beta}{\Delta_s} \end{cases}. \tag{17}$$

Finally, if we assume that $\Delta_s = 1$, it gives

$$int_L^f(\beta) = \frac{\beta}{\dot{u} \cdot \sin\beta} int_L(\beta), \tag{18}$$

where

$$int_L(\beta) = \begin{cases} \frac{1}{\Delta_\beta}\left(1 - \frac{|\beta|}{\Delta_\beta}\right) & \text{for} \quad |\beta| \le \Delta_\beta \\ 0 & \text{for} \quad |\beta| \ge \Delta_\beta \end{cases}. \tag{19}$$

In consequence, returning to the formula (9), we obtain

$$\breve{\mu}(x,y) = \frac{1}{2}\int_0^{2\pi}\int_{-\beta_m}^{\beta_m} p^f\left(\beta,\alpha^f\right)\frac{\mathrm{R}_f\cos\beta}{2\dot{u}}\frac{\Delta\beta}{\sin\Delta\beta} int_L(\Delta\beta)\, d\beta d\alpha^f. \tag{20}$$

Fortunately, we can linearize relation (21) by considering expressions inside the integration, namely $\frac{\Delta\beta}{\sin\Delta\beta}$.

In the case of linear interpolation we use only line of integrals from the neighborhood of a given pixel (x, y), then $\Delta\beta \le \Delta_\beta$, and $\sin\Delta\beta \approxeq \Delta\beta$. Additionally, it is possible to omit the term $\frac{\mathrm{R}_f\cos\beta}{2\dot{u}}$ taking into account the fact that each projection value $p^f\left(\beta,\alpha^f\right)$ has its equivalent $p^f\left(-\beta,\alpha^f+\pi+2\beta\right)$, as shown in Figure 1.

Because of this we can notice that the sum of this pair of projections is proportional to $\frac{\dot{u}_1+\dot{u}_2}{4\dot{u}_1} + \frac{\dot{u}_1+\dot{u}_2}{4\dot{u}_2} = \frac{(\dot{u}_1+\dot{u}_2)^2}{4\dot{u}_1\dot{u}_2}$. This means that for $\dot{u}_1 \approxeq \dot{u}_2$ this factor is equal to 1, and finally, we can write

$$\breve{\mu}(x,y) \approxeq \frac{1}{2}\int_0^{2\pi}\int_{-\beta_m}^{\beta_m} p^f\left(\beta,\alpha^f\right) int_L(\Delta\beta)\, d\beta d\alpha^f, \tag{21}$$

which is consistent with a form of the formula of the back-projection operation for parallel beams.

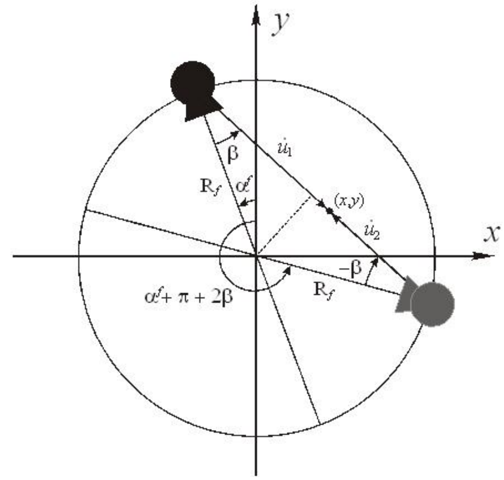

Fig. 1. Selecting complementary projection values

Moreover, if we assume that rays, i.e. integral lines defining $p^f\left(\beta,\alpha^f\right)$, from the hypothetical fan-beam geometry pass through almost the same tissues as rays from cone-beam geometry $\left(p^h\left(\beta,\alpha^h,z_k\right)\right)$, the projection values associated with these rays will be related to the corresponding path lengths through the tissues. Because of this, we can derive the correction factor by using the following relation:

$$p^f\left(\beta,\alpha^f\right) = p^h\left(\beta,\alpha^h,z_k\right) CORR = p^h\left(\beta,\alpha^h,z_k\right)\frac{R_{fd}}{\sqrt{R_{fd}^2 + z_k^2}}, \tag{22}$$

where R_{fd} is the source-to-detector distance; z_k is the transverse position on the screen where a given ray is detected.

The geometry of this method of determining the correction factor is shown in Figure 2.

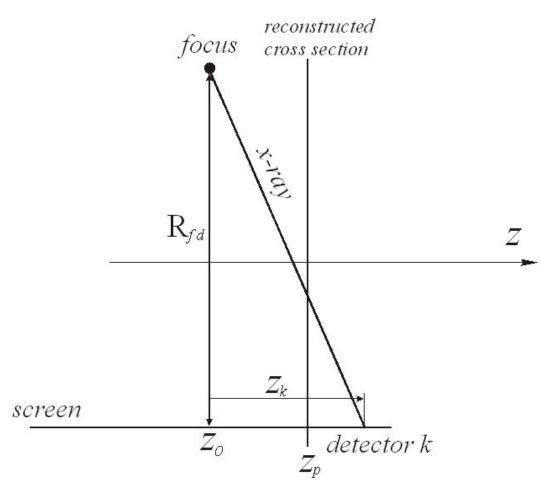

Fig. 2. The geometry of the cosine correction factor

Finally, formula (21) can be used directly to obtain a reference image for the analytical statistical iterative reconstruction algorithm presented by the formula (2), which was originally formulated for parallel beam scanner geometry, as follows

$$\breve{\mu}(x,y) \approxeq \frac{1}{2}\int\limits_{0}^{2\pi}\int\limits_{-\beta_m}^{\beta_m} CORR{\cdot}p^h\left(\beta,\alpha^h,z_k\right) int_L\left(\Delta\beta\right) d\beta d\alpha^f. \tag{23}$$

III. Experimental results

In our computer simulations, we have used projections obtained from a helical scanner Somatom Definition AS+ (Siemens Healthcare), with the following paremeters: reference tube potentia 120kVp and quality reference effective 200mAs, $R_{fd} = 1085.6mm$ (SDD - Source-to-Detector Distance); $\mathrm{R}_f = 595\mathrm{mm}$ (SOD - Source-to-AOR Distance); number of views per rotation $\Psi = 1152$; number of pixels in detector panel 736; detector dimensions $1.09\mathrm{mm} \times 1.28\mathrm{mm}$. During the experiments, the size of the processed image was fixed at 512×512 pixels. The matrix of the coefficients $h_{\Delta i,\Delta j}$ were precomputed before the reconstruction process was started, and these coefficients were fixed for the subsequent processing. The image obtained after back-projection operation was then subjected to a process of reconstruction (optimization) using an iterative procedure. The starting point of this procedure was choosen as a result of using a reconstruction FBP algorithm. It is worth noting that our reconstruction procedure was performed without any regularization regarding the objective function described by (2).

View of the reconstructed images after 30000 iterations are presented (Table 3(a). For comparison, the image reconstructed by a standard FBP reconstruction method (Table 3(b)) is also presented.

IV. Conclusion

We have shown in this paper fully feasible statistical reconstruction algorithm for helical cone-beam scanner. It is proved that this statistical approach, originally formulated for parallel beam geometry, can be adapted for heica cone-beam geometry, without any filtration and any rebinning. Simulations have been conducted, which prove that our reconstruction method can be very fast (first of all thanks to the use of FFT algorithms) and gives satisfactory results with suppressed noise, without introducing any additional regularization term, using only an early stopping regularization strategy.

(a)

(b)

Fig. 3. View of the images reconstructed image using the standard FBP (a); reconstructed image using the method described in this paper after 30000 iterations (b) ($C = 45$, $W = 600$)

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